

Small systems of Diophantine equations which have only very large integer solutions

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Abstract. Let $E_n = \{x_i = 1, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$. There is an algorithm that for every computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ returns a positive integer $m(f)$, for which a second algorithm accepts on the input f and any integer $n \geq m(f)$, and returns a system $S \subseteq E_n$ such that S has infinitely many integer solutions and each integer tuple (x_1, \dots, x_n) that solves S satisfies $x_1 = f(n)$. For each integer $n \geq 12$ we construct a system $S \subseteq E_n$ such that S has infinitely many integer solutions and they all belong to $\mathbb{Z}^n \setminus [-2^{2^{n-1}}, 2^{2^{n-1}}]^n$.

Key words and phrases: computable function, computable upper bound for the heights of integer (rational) solutions of a Diophantine equation, Davis-Putnam-Robinson-Matiyasevich theorem, Diophantine equation with a finite number of integer (rational) solutions, system of Diophantine equations.

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We present a general method for constructing small systems of Diophantine equations which have only very large integer solutions. Let Φ_n denote the following statement

$$\begin{aligned} & \forall x_1, \dots, x_n \in \mathbb{Z} \exists y_1, \dots, y_n \in \mathbb{Z} \\ & \left(2^{2^{n-1}} < |x_1| \implies (|x_1| < |y_1| \vee \dots \vee |x_1| < |y_n|) \right) \wedge \\ & \left(\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \implies y_i + y_j = y_k) \right) \wedge \\ & \left(\forall i, j, k \in \{1, \dots, n\} (x_i \cdot x_j = x_k \implies y_i \cdot y_j = y_k) \right) \end{aligned} \tag{1}$$

For $n \geq 2$, the bound $2^{2^{n-1}}$ cannot be decreased because for

$$(x_1, \dots, x_n) = (2^{2^{n-1}}, 2^{2^{n-2}}, 2^{2^{n-3}}, \dots, 256, 16, 4, 2)$$

the conjunction of statements (1) and (2) guarantees that

$$(y_1, \dots, y_n) = (0, \dots, 0) \vee (y_1, \dots, y_n) = (2^{2^{n-1}}, 2^{2^{n-2}}, 2^{2^{n-3}}, \dots, 256, 16, 4, 2)$$

The statement $\forall n \Phi_n$ has powerful consequences for Diophantine equations, but is still unproven, see [5]. In particular, it implies that if a Diophantine equation has only finitely many solutions in integers (non-negative integers, rationals), then their heights are bounded from above by a computable function of the degree and the coefficients of the equation. For integer solutions, this conjectural upper bound can be computed by applying equation (3) and Lemmas 2 and 7.

Observation. *For all positive integers n, m with $n \leq m$, if the statement Φ_n fails for $(x_1, \dots, x_n) \in \mathbb{Z}^n$ and $2^{2^{m-1}} < |x_1| \leq 2^{2^m}$, then the statement Φ_m fails for $(\underbrace{x_1, \dots, x_1}_{m-n+1 \text{ times}}, x_2, \dots, x_n) \in \mathbb{Z}^m$.*

By the Observation, the statement $\forall n \Phi_n$ is equivalent to the statement $\forall n \Psi_n$, where Ψ_n denote the statement

$$\begin{aligned} & \forall x_1, \dots, x_n \in \mathbb{Z} \exists y_1, \dots, y_n \in \mathbb{Z} \\ & (2^{2^{n-1}} < |x_1| = \max(|x_1|, \dots, |x_n|) \leq 2^{2^n} \implies (|x_1| < |y_1| \vee \dots \vee |x_1| < |y_n|)) \wedge \\ & (\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \implies y_i + y_j = y_k)) \wedge \\ & \forall i, j, k \in \{1, \dots, n\} (x_i \cdot x_j = x_k \implies y_i \cdot y_j = y_k) \end{aligned}$$

In contradistinction to the statements Φ_n , each true statement Ψ_n can be confirmed by a brute-force search in a finite amount of time.

The statement

$$\begin{aligned} & \forall n \forall x_1, \dots, x_n \in \mathbb{Z} \exists y_1, \dots, y_n \in \mathbb{Z} \\ & (2^{2^{n-1}} < |x_1| \implies |x_1| < |y_1|) \wedge \\ & (\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \implies y_i + y_j = y_k)) \wedge \\ & \forall i, j, k \in \{1, \dots, n\} (x_i \cdot x_j = x_k \implies y_i \cdot y_j = y_k) \end{aligned}$$

strengthens the statement $\forall n \Phi_n$ but is false, as we will show in the Corollary.

Let

$$E_n = \{x_i = 1, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

To each system $S \subseteq E_n$ we assign the system \widetilde{S} defined by

$$(S \setminus \{x_i = 1 : i \in \{1, \dots, n\}\}) \cup \{x_i \cdot x_j = x_j : i, j \in \{1, \dots, n\} \text{ and the equation } x_i = 1 \text{ belongs to } S\}$$

In other words, in order to obtain \widetilde{S} we remove from S each equation $x_i = 1$ and replace it by the following n equations:

$$\begin{aligned} x_i \cdot x_1 &= x_1 \\ &\dots \\ x_i \cdot x_n &= x_n \end{aligned}$$

Lemma 1. *For each system $S \subseteq E_n$*

$$\begin{aligned} \{(x_1, \dots, x_n) \in \mathbb{Z}^n : (x_1, \dots, x_n) \text{ solves } \widetilde{S}\} = \\ \{(x_1, \dots, x_n) \in \mathbb{Z}^n : (x_1, \dots, x_n) \text{ solves } S\} \cup \{(0, \dots, 0)\} \end{aligned}$$

Lemma 2. *The statement Φ_n can be equivalently stated thus: if a system $S \subseteq E_n$ has only finitely many solutions in integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $|x_1|, \dots, |x_n| \leq 2^{2^{n-1}}$.*

Proof. It follows from Lemma 1. □

Nevertheless, for each integer $n \geq 12$ there exists a system $S \subseteq E_n$ which has infinitely many integer solutions and they all belong to $\mathbb{Z}^n \setminus [-2^{2^{n-1}}, 2^{2^{n-1}}]^n$. We will prove it in Theorem 1. First we need a few lemmas.

Lemma 3. *If a positive integer n is odd and a pair (x, y) of positive integers solves the negative Pell equation $x^2 - dy^2 = -1$, then the pair*

$$\left(\frac{(x + y\sqrt{d})^n + (x - y\sqrt{d})^n}{2}, \frac{(x + y\sqrt{d})^n - (x - y\sqrt{d})^n}{2\sqrt{d}} \right)$$

consists of positive integers and solves the equation $x^2 - dy^2 = -1$.

Lemma 4. ([4, pp. 201–202, Theorem 106]) *In the domain of positive integers, all solutions to $x^2 - 5y^2 = -1$ are given by*

$$(2 + \sqrt{5})^{2k+1} = x + y\sqrt{5}$$

where k is a non-negative integer.

Lemma 5. *The pair $(2, 1)$ solves the equation $x^2 - 5y^2 = -1$. If a pair (x, y) solves the equation $x^2 - 5y^2 = -1$, then the pair $(9x + 20y, 4x + 9y)$ solves this equation too.*

Lemma 6. *Lemma 5 allows us to compute all positive integer solutions to $x^2 - 5y^2 = -1$.*

Proof. It follows from Lemma 4. Indeed, if $(2 + \sqrt{5})^{2k+1} = x + y\sqrt{5}$, then

$$\begin{aligned} (2 + \sqrt{5})^{2k+3} &= (2 + \sqrt{5})^2 \cdot (2 + \sqrt{5})^{2k+1} = \\ (9 + 4\sqrt{5}) \cdot (x + y\sqrt{5}) &= (9x + 20y) + (4x + 9y)\sqrt{5} \end{aligned}$$

□

Theorem 1. *For each integer $n \geq 12$ there exists a system $S \subseteq E_n$ such that S has infinitely many integer solutions and they all belong to $\mathbb{Z}^n \setminus [-2^{2^{n-1}}, 2^{2^{n-1}}]^n$.*

Proof. By Lemmas 4–6, the equation $u^2 - 5v^2 = -1$ has infinitely many solutions in positive integers and all these solutions can be simply computed. For a positive integer n , let $(u(n), v(n))$ denote the n -th solution to $u^2 - 5v^2 = -1$. We define S as

$$\begin{aligned} x_1 &= 1 & x_1 + x_1 &= x_2 & x_2 + x_2 &= x_3 & x_1 + x_3 &= x_4 \\ x_4 \cdot x_4 &= x_5 & x_5 \cdot x_5 &= x_6 & x_6 \cdot x_7 &= x_8 & x_8 \cdot x_8 &= x_9 \\ x_{10} \cdot x_{10} &= x_{11} & x_{11} + x_1 &= x_{12} & x_4 \cdot x_9 &= x_{12} \\ x_{12} \cdot x_{12} &= x_{13} & x_{13} \cdot x_{13} &= x_{14} & \dots & x_{n-1} \cdot x_{n-1} &= x_n \end{aligned}$$

The first 11 equations of S equivalently express that $x_{10}^2 - 5 \cdot x_8^2 = -1$ and 625 divides x_8 . The equation $x_{10}^2 - 5^9 \cdot x_7^2 = -1$ expresses the same fact. Execution of the following MuPAD code

```
x:=2:
y:=1:
for n from 2 to 313 do
u:=9*x+20*y:
v:=4*x+9*y:
if igcd(v,625)=625 then print(n) end_if:
x:=u:
y:=v:
```

```

end_for:
float(u^2+1);
float(2^(2^(12-1)));

```

returns only $n = 313$. Therefore, in the domain of positive integers, the minimal solution to $x_{10}^2 - 5^9 \cdot x_7^2 = -1$ is given by the pair $\left(x_{10} = u(313), x_7 = \frac{v(313)}{625}\right)$. Hence, if an integer tuple (x_1, \dots, x_n) solves S , then $|x_8| \geq v(313)$ and

$$x_{12} = x_{10}^2 + 1 \geq u(313)^2 + 1 > 2^{2^{12-1}}$$

The final inequality comes from the execution of the last two instructions of the code, as they display the numbers $1.263545677e783$ and $3.231700607e616$. Applying induction, we get $x_n > 2^{2^{n-1}}$. By Lemma 3 (or by [6, p. 58, Theorem 1.3.6]), the equation $x_{10}^2 - 5^9 \cdot x_7^2 = -1$ has infinitely many integer solutions. This conclusion transfers to the system S . \square

J. C. Lagarias studied the equation $x^2 - dy^2 = -1$ for $d = 5^{2n+1}$, where $n = 0, 1, 2, 3, \dots$. His theorem says that for these values of d , the least integer solution grows exponentially with d , see [2, Appendix A].

The next theorem generalizes Theorem 1. But first we need Lemma 7 together with introductory matter.

Let $D(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$. For the Diophantine equation $2 \cdot D(x_1, \dots, x_p) = 0$, let M denote the maximum of the absolute values of its coefficients. Let \mathcal{T} denote the family of all polynomials $W(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$ whose all coefficients belong to the interval $[-M, M]$ and $\deg(W, x_i) \leq d_i = \deg(D, x_i)$ for each $i \in \{1, \dots, p\}$. Here we consider the degrees of $W(x_1, \dots, x_p)$ and $D(x_1, \dots, x_p)$ with respect to the variable x_i . It is easy to check that

$$\text{card}(\mathcal{T}) = (2M + 1)^{(d_1 + 1) \cdot \dots \cdot (d_p + 1)} \quad (3)$$

We choose any bijection $\tau : \{p + 1, \dots, \text{card}(\mathcal{T})\} \longrightarrow \mathcal{T} \setminus \{x_1, \dots, x_p\}$. Let \mathcal{H} denote the family of all equations of the form

$$x_i = 1, x_i + x_j = x_k, x_i \cdot x_j = x_k \quad (i, j, k \in \{1, \dots, \text{card}(\mathcal{T})\})$$

which are polynomial identities in $\mathbb{Z}[x_1, \dots, x_p]$ if

$$\forall s \in \{p + 1, \dots, \text{card}(\mathcal{T})\} \quad x_s = \tau(s)$$

There is a unique $q \in \{p+1, \dots, \text{card}(\mathcal{T})\}$ such that $\tau(q) = 2 \cdot D(x_1, \dots, x_p)$. For each ring \mathbf{K} extending \mathbb{Z} the system \mathcal{H} implies $2 \cdot D(x_1, \dots, x_p) = x_q$. To see this, we observe that there exist pairwise distinct $t_0, \dots, t_m \in \mathcal{T}$ such that $m > p$ and

$$t_0 = 1 \wedge t_1 = x_1 \wedge \dots \wedge t_p = x_p \wedge t_m = 2 \cdot D(x_1, \dots, x_p) \wedge$$

$$\forall i \in \{p+1, \dots, m\} \exists j, k \in \{0, \dots, i-1\} (t_j + t_k = t_i \vee t_i + t_k = t_j \vee t_j \cdot t_k = t_i)$$

For each ring \mathbf{K} extending \mathbb{Z} and for each $x_1, \dots, x_p \in \mathbf{K}$ there exists a unique tuple $(x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}) \in \mathbf{K}^{\text{card}(\mathcal{T})-p}$ such that the tuple $(x_1, \dots, x_p, x_{p+1}, \dots, x_{\text{card}(\mathcal{T})})$ solves the system \mathcal{H} . The sought elements $x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}$ are given by the formula

$$\forall s \in \{p+1, \dots, \text{card}(\mathcal{T})\} \quad x_s = \tau(s)(x_1, \dots, x_p)$$

Lemma 7. *The system $\mathcal{H} \cup \{x_q + x_q = x_q\}$ can be simply computed. For each ring \mathbf{K} extending \mathbb{Z} , the equation $D(x_1, \dots, x_p) = 0$ is equivalent to the system $\mathcal{H} \cup \{x_q + x_q = x_q\} \subseteq E_{\text{card}(\mathcal{T})}$. Formally, this equivalence can be written as*

$$\forall x_1, \dots, x_p \in \mathbf{K} \left(D(x_1, \dots, x_p) = 0 \iff \exists x_{p+1}, \dots, x_{\text{card}(\mathcal{T})} \in \mathbf{K} \right.$$

$$\left. (x_1, \dots, x_p, x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}) \text{ solves the system } \mathcal{H} \cup \{x_q + x_q = x_q\} \right)$$

For each ring \mathbf{K} extending \mathbb{Z} and for each $x_1, \dots, x_p \in \mathbf{K}$ with $D(x_1, \dots, x_p) = 0$ there exists a unique tuple $(x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}) \in \mathbf{K}^{\text{card}(\mathcal{T})-p}$ such that the tuple $(x_1, \dots, x_p, x_{p+1}, \dots, x_{\text{card}(\mathcal{T})})$ solves the system $\mathcal{H} \cup \{x_q + x_q = x_q\}$. Hence, for each ring \mathbf{K} extending \mathbb{Z} the equation $D(x_1, \dots, x_p) = 0$ has the same number of solutions as the system $\mathcal{H} \cup \{x_q + x_q = x_q\}$.

Putting $M = M/2$ we obtain new families \mathcal{T} and \mathcal{H} . There is a unique $q \in \{1, \dots, \text{card}(\mathcal{T})\}$ such that

$$(q \in \{1, \dots, p\} \wedge x_q = D(x_1, \dots, x_p)) \vee$$

$$(q \in \{p+1, \dots, \text{card}(\mathcal{T})\} \wedge \tau(q) = D(x_1, \dots, x_p))$$

The new system $\mathcal{H} \cup \{x_q + x_q = x_q\}$ is equivalent to $D(x_1, \dots, x_p) = 0$ and can be simply computed.

The Davis-Putnam-Robinson-Matiyasevich theorem states that every recursively enumerable set $\mathcal{M} \subseteq \mathbb{N}^n$ has a Diophantine representation, that is

$$(a_1, \dots, a_n) \in \mathcal{M} \iff \exists x_1, \dots, x_m \in \mathbb{N} \quad W(a_1, \dots, a_n, x_1, \dots, x_m) = 0$$

for some polynomial W with integer coefficients, see [3] and [1]. The polynomial W can be computed, if we know a Turing machine M such that, for all $(a_1, \dots, a_n) \in \mathbb{N}^n$, M halts on (a_1, \dots, a_n) if and only if $(a_1, \dots, a_n) \in \mathcal{M}$, see [3] and [1].

Theorem 2. *There is an algorithm that for every computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ returns a positive integer $m(f)$, for which a second algorithm accepts on the input f and any integer $n \geq m(f)$, and returns a system $S \subseteq E_n$ such that S has infinitely many integer solutions and each integer tuple (x_1, \dots, x_n) that solves S satisfies $x_1 = f(n)$.*

Proof. By the Davis-Putnam-Robinson-Matiyasevich theorem, the function f has a Diophantine representation. It means that there is a polynomial $W(x_1, x_2, x_3, \dots, x_r)$ with integer coefficients such that for each non-negative integers x_1, x_2 ,

$$x_1 = f(x_2) \iff \exists x_3, \dots, x_r \in \mathbb{N} \quad W(x_1, x_2, x_3, \dots, x_r) = 0 \quad (\text{E1})$$

By the equivalence (E1) and Lagrange's four-square theorem, for each integers x_1, x_2 , the conjunction $(x_2 \geq 0) \wedge (x_1 = f(x_2))$ holds true if and only if there exist integers $a, b, c, d, \alpha, \beta, \gamma, \delta, x_3, x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, \dots, x_r, x_{r,1}, x_{r,2}, x_{r,3}, x_{r,4}$ such that

$$\begin{aligned} & W^2(x_1, x_2, x_3, \dots, x_r) + (x_1 - a^2 - b^2 - c^2 - d^2)^2 + (x_2 - \alpha^2 - \beta^2 - \gamma^2 - \delta^2)^2 + \\ & (x_3 - x_{3,1}^2 - x_{3,2}^2 - x_{3,3}^2 - x_{3,4}^2)^2 + \dots + (x_r - x_{r,1}^2 - x_{r,2}^2 - x_{r,3}^2 - x_{r,4}^2)^2 = 0 \end{aligned}$$

By Lemma 7, there is an integer $s \geq 3$ such that for each integers x_1, x_2 ,

$$(x_2 \geq 0 \wedge x_1 = f(x_2)) \iff \exists x_3, \dots, x_s \in \mathbb{Z} \quad \Psi(x_1, x_2, x_3, \dots, x_s) \quad (\text{E2})$$

where the formula $\Psi(x_1, x_2, x_3, \dots, x_s)$ is algorithmically determined as a conjunction of formulae of the form $x_i = 1$, $x_i + x_j = x_k$, $x_i \cdot x_j = x_k$ ($i, j, k \in \{1, \dots, s\}$). Let $m(f) = 8 + 2s$, and let $[\cdot]$ denote the integer part function. For each integer $n \geq m(f)$,

$$n - \left\lfloor \frac{n}{2} \right\rfloor - 4 - s \geq m(f) - \left\lfloor \frac{m(f)}{2} \right\rfloor - 4 - s \geq m(f) - \frac{m(f)}{2} - 4 - s = 0$$

Let S denote the following system

$$\left\{ \begin{array}{l} \text{all equations occurring in } \Psi(x_1, x_2, \dots, x_s) \\ n - \left\lceil \frac{n}{2} \right\rceil - 4 - s \text{ equations of the form } z_i = 1 \\ \begin{array}{rcl} t_1 & = & 1 \\ t_1 + t_1 & = & t_2 \\ t_2 + t_1 & = & t_3 \\ & \dots & \\ t_{\lceil \frac{n}{2} \rceil - 1} + t_1 & = & t_{\lceil \frac{n}{2} \rceil} \\ t_{\lceil \frac{n}{2} \rceil} + t_{\lceil \frac{n}{2} \rceil} & = & w \\ w + y & = & x_2 \\ y + y & = & y \text{ (if } n \text{ is even)} \\ y & = & 1 \text{ (if } n \text{ is odd)} \\ u + u & = & v \end{array} \end{array} \right.$$

with n variables. By the equivalence (E2), the system S is consistent over \mathbb{Z} . The equation $u + u = v$ guarantees that S has infinitely many integer solutions. If an integer n -tuple $(x_1, x_2, \dots, x_s, \dots, w, y, u, v)$ solves S , then by the equivalence (E2),

$$x_1 = f(x_2) = f(w + y) = f\left(2 \cdot \left\lceil \frac{n}{2} \right\rceil + y\right) = f(n)$$

□

Corollary. *There is an algorithm that for every computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ returns a positive integer $m(f)$, for which a second algorithm accepts on the input f and any integer $n \geq m(f)$, and returns an integer tuple (x_1, \dots, x_n) for which $x_1 = f(n)$ and*

(4) *for each integers y_1, \dots, y_n the conjunction*

$$\left(\forall i \in \{1, \dots, n\} (x_i = 1 \implies y_i = 1) \right) \wedge$$

$$\left(\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \implies y_i + y_j = y_k) \right) \wedge$$

$$\forall i, j, k \in \{1, \dots, n\} (x_i \cdot x_j = x_k \implies y_i \cdot y_j = y_k)$$

implies that $x_1 = y_1$.

Proof. Let \leq_n denote the order on \mathbb{Z}^n which ranks the tuples (x_1, \dots, x_n) first according to $\max(|x_1|, \dots, |x_n|)$ and then lexicographically. The ordered set

(\mathbb{Z}^n, \leq_n) is isomorphic to (\mathbb{N}, \leq) . To find an integer tuple (x_1, \dots, x_n) , we solve the system S by performing the brute-force search in the order \leq_n .

□

If $n \geq 2$, then the tuple

$$(x_1, \dots, x_n) = (2^{2^{n-2}}, 2^{2^{n-3}}, \dots, 256, 16, 4, 2, 1)$$

has property (4). Unfortunately, we do not know any explicitly given integers x_1, \dots, x_n with property (4) and $|x_1| > 2^{2^{n-2}}$.

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